Solution of the Lindblad Equation in the Kraus Representation

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The so-called Lindblad equation, a typical master equation describing the dissipative quantum dynamics, is shown to be solvable for finite-level systems in a compact form without resort to writing it down as a set of equations among matrix elements. The solution is then naturally given in an operator form, known as the Kraus representation. Following a few simple examples, the general applicability of the method is clarified.

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I. INTRODUCTION

A quantum system in interaction with an environment, that is a larger system with a continuous number of degrees of freedom, displays an irreversible, dissipative, and decohering dynamics, which is described by a master equation [1, 2, 3, 4, 5, 6, 7, 8]. The ability of solving the master equations plays a considerable role in different physical contexts both from practical and fundamental points of view. It enables us to discuss the thermalization processes and to highlight the problem of individualizing the quantum–classical boundary [9]. Master equations are also necessary for the realistic description of many quantum systems exploited in quantum information processing [10, 11, 12, 13, 14, 15, 16, 17].

The so-called Lindblad equation [18] is the most general form of linear and Markovian master equation preserving the probability and the positivity of the density operator. Although the Lindblad equation can be derived, under appropriate approximations or in some limits, from a microscopic model consisting of the quantum system and the environment after tracing out the latter degrees of freedom [1, 2, 3, 4, 5, 6, 7, 8], one can even start with it as a phenomenological equation [19].

Generally speaking, solving master equations requires the resolution of a set of differential equations among the matrix elements of the density operator with respect to a specific basis, usually the eigenstates of the system Hamiltonian. Nevertheless, it is generally a cumbersome task to list up all of the N^2 equations, N being the dimension of the density matrix (the number of states), and if one selects the basis carelessly, many of the equations look coupled to each other, even though they are not essentially so on the appropriate basis.

Moreover, even in those lucky cases wherein one can find the solution to the set of the differential equations, the density operator of the system is given as a collection of the matrix elements with respect to the specific basis, instead of being expressed in an operator form. It is known that the evolution of the density operator is given in the Kraus representation [20], but it is not easy to construct it from the set of the matrix elements. Over the last decade, several efforts have been done in this direction and several procedures for constructing operator solutions to master equations under appropriate conditions have been proposed [21, 22].

In this article, we present a procedure for solving Markovian master equations directly in an operator form in the Kraus representation, instead of giving it by specifying its matrix elements. If one works with the matrix elements, one needs to write down an $N^2 \times N^2$ matrix which couples the N^2 equations for the matrix elements, and then, realizes that many of them are not really coupled to each other (for relatively simple systems). In the approach discussed here, on the other hand, it is not required to write such a sparse matrix, where most of its elements are zero, but one can realize the minimal set of the necessary equations by looking at the relevant part of the dissipator of the master equation, associated with the "quantum jumps" [23, 24].

The approach is initially sketched in Sec. II with two simple examples, few-level systems at zero and finite temperatures, showing the idea and its powerfulness. Then, we discuss its general applicability in Sec. III and conclude the article in Sec. IV. In Appendices A and B, we summarize a derivation of master equations, giving possible connections between the master equations discussed in the main text and microscopic Hamiltonians, and present some details about the solution to the second example considered in Sec. II B.

II. SIMPLE EXAMPLES

In this section, the resolution of two master equations describing the dynamics of two physical systems is presented. The first example we discuss refers to a thermal

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reservoir at zero temperature whereas in the second one a finite-temperature reservoir ($T \neq 0$) is considered. We successfully solve the respective master equations building up in both cases solutions in the operator form known as the Kraus representation instead of expressing them in terms of a collection of matrix elements. These two examples aim at illustrating the "key ingredients" and the basic steps of the general procedure we are going to present in Sec. III.

A. A Three-Level System at the Zero Temperature

Consider a three-level system at the zero temperature and suppose its dynamics is described by the following master equation

$$\begin{aligned}\dot{\rho}(t) = &-i[H, \rho(t)] \\ &- \gamma \left(\frac{1}{2}\{|1\rangle\langle 1|, \rho(t)\} - |0\rangle\langle 1|\rho(t)|1\rangle\langle 0| \right) \\ &- \gamma \left(\frac{1}{2}\{|2\rangle\langle 2|, \rho(t)\} - |1\rangle\langle 2|\rho(t)|2\rangle\langle 1| \right), \end{aligned} \tag{2.1}$$

where the states $|i\rangle$ $(i = 0, 1, 2)$ diagonalize the Hamiltonian H,

$$H = \sum_{i=0,1,2} E_i |i\rangle\langle i|. \tag{2.2}$$

This master equation can describe the dissipative dynamics in the triplet sector of two qubits at the zero temperature, as shown in Appendix A 1. In order to solve the master equation (2.1), observe that it can be rewritten in the following form:

$$\dot{\rho}(t) = A\rho(t) + \rho(t)A^\dagger + \gamma \sum_{i=0,1} B_i \rho(t) B_i^\dagger \tag{2.3}$$

with

$$A = -iH - \frac{\gamma}{2}\Big(|1\rangle\langle 1| + |2\rangle\langle 2| \Big), \tag{2.4a}$$

$$B_0 = |0\rangle\langle 1|, \quad B_1 = |1\rangle\langle 2|. \tag{2.4b}$$

This rewriting of the master equation naturally suggests one to introduce the transformation

$$\rho(t) = e^{At} \rho_{\mathrm{I}}(t) e^{A^\dagger t}. \tag{2.5}$$

It is indeed easy to convince oneself that $\rho_{\mathrm{I}}(t)$ evolves according to the equation

$$\dot{\rho}_{\mathrm{I}}(t) = \gamma \sum_{i=0,1} B_i(t) \rho_{\mathrm{I}}(t) B_i^\dagger(t), \quad B_i(t) = e^{-At} B_i e^{At}. \tag{2.6}$$

On the other hand, explicitly evaluating the operators $B_i(t)$, we obtain

$$B_0(t) = B_0 e^{-i(E_1 - E_0)t} e^{-\gamma t/2}, \quad B_1(t) = B_1 e^{-i(E_2 - E_1)t}, \tag{2.7}$$

and then (2.6) is expressed as

$$\dot{\rho}_{\mathrm{I}}(t) = \gamma B_0 \rho_{\mathrm{I}}(t) B_0^\dagger e^{-\gamma t} + \gamma B_1 \rho_{\mathrm{I}}(t) B_1^\dagger. \tag{2.8}$$

It is worth noting that B_i's are essentially ladder operators satisfying the following relations,

$$B_0^2 = B_1^2 = B_1 B_0 = 0, \quad B_0 B_1 = |0\rangle\langle 2| \neq 0, \tag{2.9}$$

so that we get

$$B_0 \dot{\rho}_{\mathrm{I}}(t) B_0^\dagger = \gamma B_0 B_1 \rho_{\mathrm{I}}(t) B_1^\dagger B_0^\dagger, \quad B_1 \dot{\rho}_{\mathrm{I}}(t) B_1^\dagger = 0. \tag{2.10}$$

These equations are easily integrated to obtain

$$B_1 \rho_{\mathrm{I}}(t) B_1^\dagger = B_1 \rho_{\mathrm{I}}(0) B_1^\dagger = B_1 \rho(0) B_1^\dagger, \tag{2.11a}$$

$$B_0 \rho_{\mathrm{I}}(t) B_0^\dagger = B_0 \rho(0) B_0^\dagger + \gamma t B_0 B_1 \rho(0) B_1^\dagger B_0^\dagger. \tag{2.11b}$$

It is now straightforward to deduce the expression of $\rho_{\mathrm{I}}(t)$, by simply integrating the right-hand side of (2.8) with (2.11) taken into account,

$$\begin{aligned}\rho_{\mathrm{I}}(t) = \rho(0) &+ (1 - e^{-\gamma t}) B_0 \rho(0) B_0^\dagger + \gamma t B_1 \rho(0) B_1^\dagger \\ &+ (1 - e^{-\gamma t} - \gamma t e^{-\gamma t}) B_0 B_1 \rho(0) B_1^\dagger B_0^\dagger. \end{aligned} \tag{2.12}$$

Finally, coming back to the original picture through the operator

$$\begin{aligned}e^{At} = e^{-iE_2 t} e^{-\gamma t/2} |2\rangle\langle 2| + e^{-iE_1 t} e^{-\gamma t/2} |1\rangle\langle 1| \\ + e^{-iE_0 t} |0\rangle\langle 0|, \end{aligned} \tag{2.13}$$

we get the explicit expression of the density operator at an arbitrary time $t > 0$,

$$\begin{aligned}\rho(t) = &\, e^{At} \rho(0) e^{A^\dagger t} \\ &+ (1 - e^{-\gamma t}) B_0 \rho(0) B_0^\dagger + \gamma t e^{-\gamma t} B_1 \rho(0) B_1^\dagger \\ &+ (1 - e^{-\gamma t} - \gamma t e^{-\gamma t}) B_0 B_1 \rho(0) B_1^\dagger B_0^\dagger. \end{aligned} \tag{2.14}$$

We wish to stress that the master equation has been solved in the operator form and its solution naturally falls into the Kraus representation. As expected, the procedure presented in this section does not take into account at all the set of the equations among the matrix elements, which, in this case, involves a 9×9 matrix (or at least, two equations for the diagonal elements and three equations for the off-diagonal elements). We only need two coupled equations (2.10). Therefore, a great simplification has been achieved through the present approach. A key element is to introduce the transformation (2.5) and rewrite the master equation into the form (2.6). The number of the necessary equations (two in this case) relies on the structure of the right-hand side of (2.6), related to the so-called "quantum jumps" [23, 24].

In the case under scrutiny, the thermal reservoir is at the zero temperature, which results in the vanishing derivative in the second equation in (2.10). This feature makes easier the resolution of the set of the equations in

(2.10): in fact, one can proceed one by one, solve the second equation first and then the first one after plugging the solution of the second into the first. We stress that this feature is special for zero-temperature cases and reflects the fact that no transitions into the higher energy levels from the lower energy ones occur at the zero temperature. Similar structures are found in Refs. [22, 24].

B. A Two-Level System at a Finite Temperature

The second example is a single two-level system at a finite temperature, described by the master equation

$$\begin{aligned}\dot{\rho}(t) = &-\frac{i}{2}\Omega[\sigma_z, \rho(t)] \\ &- \gamma_+ \left(\frac{1}{2}\{\sigma_+\sigma_-, \rho(t)\} - \sigma_-\rho(t)\sigma_+\right) \\ &- \gamma_- \left(\frac{1}{2}\{\sigma_-\sigma_+, \rho(t)\} - \sigma_+\rho(t)\sigma_-\right). \end{aligned} \tag{2.15}$$

σ_z is a spin operator and $\sigma_\pm$ are ladder operators. The first term on the right-hand side describes the unitary evolution, the second the spontaneous and the induced decays from the upper to the lower level, and the third the induced transition from the lower to the upper. This master equation is derived from a spin-boson model: see Appendix A 2.

As in the previous example, we arrange the master equation into

$$\dot{\rho}_{\rm I}(t) = \gamma_+\sigma_-\rho_{\rm I}(t)\sigma_+ e^{-\gamma t} + \gamma_-\sigma_+\rho_{\rm I}(t)\sigma_- e^{\gamma t}, \tag{2.16}$$

where $\rho_{\rm I}(t)$ is defined by (2.5) with

$$A = -\frac{1}{4}\gamma^\beta - \frac{1}{4}(\gamma + 2i\Omega)\sigma_z, \tag{2.17a}$$

$$\gamma^\beta = \gamma_+ + \gamma_-, \quad \gamma = \gamma_+ - \gamma_-. \tag{2.17b}$$

Starting from (2.16) and taking into account that $\hat{\sigma}_\pm^2 = 0$, it is immediate to reach the following equations:

$$\begin{cases} \sigma_+\dot{\rho}_{\rm I}(t)\sigma_- = \gamma_+\sigma_+\sigma_-\rho_{\rm I}(t)\sigma_+\sigma_- e^{-\gamma t}, \\ \sigma_-\dot{\rho}_{\rm I}(t)\sigma_+ = \gamma_-\sigma_-\sigma_+\rho_{\rm I}(t)\sigma_-\sigma_+ e^{\gamma t}. \end{cases} \tag{2.18}$$

In contrast to the previous example, i.e. (2.10) for the zero-temperature case, none of the derivatives in (2.18) is vanishing in general. The solution to this set of equations is given in (B8). Now, by integrating (2.16) with the solution (B8) taken into account and by going back to the original picture, we obtain the solution to the master equation (2.15):

$$\begin{aligned}\rho(t) = &\frac{1}{4}\rho(0)(1 + e^{-\gamma^\beta t} + 2e^{-\gamma^\beta t/2}\cos\Omega t) \\ &+ \frac{1}{4}\sigma_z\rho(0)\sigma_z(1 + e^{-\gamma^\beta t} - 2e^{-\gamma^\beta t/2}\cos\Omega t) \\ &- \frac{1}{4}\rho(0)\sigma_z\left(\frac{\gamma}{\gamma^\beta}(1 - e^{-\gamma^\beta t}) - 2ie^{-\gamma^\beta t/2}\sin\Omega t\right) \\ &- \frac{1}{4}\sigma_z\rho(0)\left(\frac{\gamma}{\gamma^\beta}(1 - e^{-\gamma^\beta t}) + 2ie^{-\gamma^\beta t/2}\sin\Omega t\right) \\ &+ (1 - e^{-\gamma^\beta t})\left(\frac{\gamma_+}{\gamma^\beta}\sigma_-\rho(0)\sigma_+ + \frac{\gamma_-}{\gamma^\beta}\sigma_+\rho(0)\sigma_-\right). \end{aligned} \tag{2.19}$$

At this point, it is worth observing that the above examples (2.1) and (2.15) keep the general characteristics of the Lindblad master equation, and this suggests the possibility of applying the procedure to more general cases, which is the subject of the next section.

III. APPLICATION TO MORE GENERAL MASTER EQUATIONS

So far, we have solved two master equations for finite-level systems, one at zero and the other at finite temperature. Our approach essentially consists in writing down a set of equations for suitable operators through which it is possible to reconstruct the complete solution of the master equation. In other words, by means of the resolution of the partial problems given in (2.10) and (2.18), it is possible to give the complete resolution of (2.1) and (2.15), respectively. In this section, we are going to analyze a more general situation and identify a set of operators which allow us to reproduce the basic steps of the approach previously exploited. To better understand this point, consider the following master equation

$$\begin{aligned}\dot{\rho}(t) = &-i[H, \rho(t)] \\ &- \sum_{m,n\in{\rm I}} \gamma_{mn}\left(\frac{1}{2}\{X_{mn}^\dagger X_{mn}, \rho(t)\} - X_{mn}\rho(t)X_{mn}^\dagger\right), \end{aligned} \tag{3.1a}$$

where

$$H = \sum_n E_n|n\rangle\langle n| = \sum_n E_n X_{nn}, \quad X_{mn} = |m\rangle\langle n|. \tag{3.1b}$$

The summations over m and n in the dissipator are taken only over the relevant eigenstates to the interaction with the reservoir, the set of which is symbolically denoted by I. This master equation is not the most general master equation of the Lindblad form, but is the most general one derived from the microscopic Hamiltonian (A1) under the Born–Markov approximation in the weak-coupling regime [1, 2, 3, 4, 5, 6, 7, 8] or in the van Hove limit [25, 26, 27], as far as neither the energy $H^{(0)}$ nor the energy differences ω_{mn} in I is degenerated (see Appendix A). These limitations are just for simplicity, and the procedure works for even more general cases (for finite-level systems), at least in principle.

Following the procedure presented in the previous section, rearrange the Liouvillian just in the same form as in (2.3),

$$\dot{\rho}(t) = A\rho(t) + \rho(t)A^\dagger + \sum_{m,n\in{\rm I}} \gamma_{mn}X_{mn}\rho(t)X_{mn}^\dagger, \tag{3.2}$$

where

$$A = -iH - \frac{1}{2}\sum_{m,n\in\mathrm{I}} \gamma_{mn} X_{mn}^\dagger X_{mn} = -i\sum_n \left(E_n - \frac{i}{2}\sum_{m\in\mathrm{I}} \gamma_{mn} \right) X_{nn}. \tag{3.3}$$

The transformed operators $X_{mn}(t)$ are easily calculated to be

$$X_{mn}(t) = e^{-At} X_{mn} e^{At} = \exp\left(i(E_m - E_n)t + \frac{1}{2}\sum_{\ell\in\mathrm{I}} (\gamma_{\ell m} - \gamma_{\ell n})t \right) X_{mn} \tag{3.4}$$

and the transformed density matrix $\rho_{\mathrm{I}}(t)$ satisfies

$$\dot\rho_{\mathrm{I}}(t) = \sum_{m,n\in\mathrm{I}} \gamma_{mn} X_{mn}(t)\rho_{\mathrm{I}}(t) X_{mn}^\dagger(t) = \sum_{m,n\in\mathrm{I}} \gamma_{mn} e^{\sum_{\ell\in\mathrm{I}}(\gamma_{\ell m} - \gamma_{\ell n})t} X_{mn}\rho_{\mathrm{I}}(t) X_{mn}^\dagger. \tag{3.5}$$

From this equation, we get

$$\frac{d}{dt}\left(e^{\sum_{\ell\in\mathrm{I}}(\gamma_{\ell m} - \gamma_{\ell n})t} X_{mn}\rho_{\mathrm{I}}(t) X_{mn}^\dagger \right) = \sum_{n'\in\mathrm{I}} (\Gamma_m)_{nn'} \left(e^{\sum_{\ell\in\mathrm{I}}(\gamma_{\ell m} - \gamma_{\ell n'})t} X_{mn'}\rho_{\mathrm{I}}(t) X_{mn'}^\dagger \right), \tag{3.6}$$

where

$$(\Gamma_m)_{nn'} = \delta_{nn'} \sum_{\ell\in\mathrm{I}} (\gamma_{\ell m} - \gamma_{\ell n}) + \gamma_{nn'}, \tag{3.7}$$

and by plugging its solution into (3.5), we obtain

$$\rho_{\mathrm{I}}(t) = \rho(0) + \sum_{m,n\in\mathrm{I}} \gamma_{mn} \sum_{n'} \left(\int_0^t e^{\Gamma_m s} ds \right)_{nn'} X_{mn'}\rho(0) X_{mn'}^\dagger, \tag{3.8a}$$

which, in the case wherein the inverse Γ_m^{-1} exists, reduces to

$$\rho_{\mathrm{I}}(t) = \rho(0) + \sum_{m,n,n'\in\mathrm{I}} \gamma_{mn} [\Gamma_m^{-1}(e^{\Gamma_m t} - 1)]_{nn'} X_{mn'}\rho(0) X_{mn'}^\dagger. \tag{3.8b}$$

Since the operator A is already diagonal and therefore

$$e^{At} = \sum_n e^{-i\tilde E_n t} X_{nn}, \quad e^{A^\dagger t} = \sum_n e^{i\tilde E_n^* t} X_{nn}, \tag{3.9a}$$

$$\tilde E_n = E_n - \frac{i}{2}\sum_{m\in\mathrm{I}} \gamma_{mn}, \tag{3.9b}$$

the solution $\rho(t) = e^{At}\rho_{\mathrm{I}}(t)e^{A^\dagger t}$ easily follows.

From the above procedure for finding the solution to the master equation of the Lindblad form, it is clear that the remaining problem is to calculate the integral of the exponential of the matrix Γ_m defined in (3.7). Such an integral always exists for bounded matrices Γ_m's. Then, the range of effectiveness of this procedure would be evident. We understand that this procedure is quite useful and effective whenever there are only a few channels through which the system interacts with the reservoir, because in such a case the dimension of the matrix Γ_m, which is nothing but the number of levels coupled with the reservoir, is small enough to be practically treated, irrespectively of the dimension of the system itself, which may be very large. On the contrary, as the number of levels interacting with the reservoir increases, the dimension of the matrix Γ_m increases linearly and the procedure here expounded becomes less effective.

The above form of the solution (3.8) is very expressive since it separates the part of the evolution associated with the "quantum jumps," $X_{mn'}\rho(0)X_{mn'}^\dagger$, and the "no-jump processes," corresponding to $e^{At}\rho(0)e^{A^\dagger t}$ [23, 24]. It clearly corresponds to the quantum-jump formula given in Refs. [23, 24], but it should be noted that the multiple integrals and the infinite sum of them appearing in such a quantum-jump formal formula, which are usually difficult to carry out directly, are already done in the explicit solution expressed by (3.8). Moreover, it is worth stressing that the jump contribution is evaluated somehow block by block, hence individualizing a sort of invariant blocks in the Liouville space. In fact, the matrix $e^{\Gamma_m t}$ is responsible for the evolution of a set of the contractions of the density operator, that is $(X_{m0}\rho X_{m0}^\dagger, X_{m1}\rho X_{m1}^\dagger, \ldots, X_{m(N-1)}\rho X_{m(N-1)}^\dagger)$. This is well visible for instance from (3.6).

The master equation (3.1) is derived under the assumption that there is no degeneracy in the energy difference $\omega_{mn} = E_m^{(0)} - E_n^{(0)}$. Possible degeneracies in the energy difference would result in a slightly different form of master equation. In such a case, roughly speaking, the operator X_{mn} in (3.1) has to be replaced with the sum of operators with the same energy difference and therefore the Liouvillian would get additional off-diagonal terms. The example (A14) bears such terms (the second term of the generator). It is worth stressing that the procedure presented here equally applies to this case as demonstrated in Appendix B, and further, it works even for the most general master equation of the Lindblad form in principle (as far as the number of the decay channels is finite).

A. A Three-Level System at the Zero Temperature

Finally, it is instructive to consider again the two examples solved in Sec. II, showing how those solutions are reproduced through the general formula (3.8). To this end, it is enough to find the relevant Γ_m matrices de-

fined in (3.7). For the first example (2.1), only two decay constants are nonvanishing, $\gamma_{01} = \gamma_{12} = \gamma$, and there are only two relevant matrices, $\Gamma_m = \gamma\hat{\Gamma}_m$ $(m = 0, 1)$:

$$\hat{\Gamma}_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \hat{\Gamma}_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}. \tag{3.10}$$

They are neither invertible nor diagonalizable. The exponentials of these matrices are calculated to be

$$e^{\Gamma_1 t} = \begin{pmatrix} e^{\gamma t} & -1 + e^{\gamma t} & -1 - \gamma t + e^{\gamma t} \\ 0 & 1 & \gamma t \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.11a}$$

$$e^{\Gamma_0 t} = \begin{pmatrix} 1 & 1 - e^{-\gamma t} & 1 - e^{-\gamma t} - \gamma t e^{-\gamma t} \\ 0 & e^{-\gamma t} & \gamma t e^{-\gamma t} \\ 0 & 0 & e^{-\gamma t} \end{pmatrix}, \tag{3.11b}$$

where $(e^{\Gamma_0 t})_{11}$, $(e^{\Gamma_0 t})_{12}$, and $(e^{\Gamma_1 t})_{22}$ elements contribute to the solution (3.8a), which reproduces (2.12).

B. A Two-Level System at a Finite Temperature

For the second example (2.15), we have

$$\Gamma_0 = \begin{pmatrix} 0 & \gamma_+ \\ \gamma_- & -\gamma \end{pmatrix}, \qquad \Gamma_1 = \begin{pmatrix} \gamma & \gamma_+ \\ \gamma_- & 0 \end{pmatrix}, \tag{3.12}$$

which are invertible and

$$e^{\Gamma_0 t} = \frac{1}{\gamma^\beta} \begin{pmatrix} \gamma_- e^{-\gamma_+ t} + \gamma_+ e^{\gamma_- t} & \gamma_+ (e^{\gamma_- t} - e^{-\gamma_+ t}) \\ \gamma_- (e^{\gamma_- t} - e^{-\gamma_+ t}) & \gamma_+ e^{-\gamma_+ t} + \gamma_- e^{\gamma_- t} \end{pmatrix}, \tag{3.13a}$$

$$e^{\Gamma_1 t} = \frac{1}{\gamma^\beta} \begin{pmatrix} \gamma_+ e^{\gamma_+ t} + \gamma_- e^{-\gamma_- t} & \gamma_+ (e^{\gamma_+ t} - e^{-\gamma_- t}) \\ \gamma_- (e^{\gamma_+ t} - e^{-\gamma_- t}) & \gamma_- e^{\gamma_+ t} + \gamma_+ e^{-\gamma_- t} \end{pmatrix}. \tag{3.13b}$$

The elements $(e^{\Gamma_0 t})_{10}$, $(e^{\Gamma_0 t})_{11}$, $(e^{\Gamma_1 t})_{00}$, and $(e^{\Gamma_1 t})_{01}$ are relevant in the solution (3.8), and we obtain (2.19).

IV. SUMMARY

In this article, we have presented a general approach to the resolution of the master equations in the Lindblad form. Through this procedure, it is possible to solve the Markovian master equation in a compact form without resort to writing it down as a set of equations among the matrix elements. The solution is directly given in an operator form, known as the Kraus representation. Furthermore, one needs to solve less differential equations than for the matrix elements and does not need to write down a sparse matrix for the coupled differential equations among the matrix elements. This difference is remarkable for relatively simple systems like the first example, for which the very simple algebra of the operators involved in the master equation allows us to find the solution in a rather simple operator form. In fact, the minimal set of the relevant equations is determined by the structure of a part of the dissipator, the generator of the evolution of the transformed density operator in (3.5), which is responsible for the so-called "quantum jumps" [23, 24], and it individualizes a sort of invariant blocks in the Liouville space. The solution (3.8) to the master equation (3.1) is interesting since it clearly separates the "quantum jumps" and the "no-jump processes." A remarkable point is that, although we demonstrated the present approach with specific examples, one can solve general Markovian master equations in principle according to the same idea presented here, as far as the number of the decay channels is finite.

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APPENDIX A: DERIVATION OF THE LINDBLAD EQUATIONS

In this appendix, we wish to give possible relationships between the master equations discussed in the text and microscopic Hamiltonians. Consider a system whose Hamiltonian is diagonalized by a set of nondegenerate eigenstates,

$$H^{(0)} = \sum_n E_n^{(0)} |n\rangle\langle n| = \sum_n E_n^{(0)} X_{nn}, \tag{A1a}$$

and assume that the system interacts with a bosonic reservoir, whose annihilation and creation operators are denoted by $a_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}^\dagger$, respectively, and whose Hamilto-

nian is given by

$$H_{\rm R} = \int d^3\boldsymbol{k}\, \omega_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}}, \qquad \omega_{\boldsymbol{k}} > 0. \tag{A1b}$$

We consider the general linear interaction between the system and the bosonic reservoir,

$$H_{\rm int} = \sum_{m,n\in {\rm I}} (X_{mn}\otimes R^\dagger_{mn} + X^\dagger_{mn}\otimes R_{mn}), \tag{A1c}$$

$$R_{mn} = \int d^3\boldsymbol{k}\, h^*_{mn}(\boldsymbol{k}) a_{\boldsymbol{k}}, \quad X_{mn} = |m\rangle\langle n|, \tag{A1d}$$

where the summations over m and n are taken only over the eigenstates actually involved in the interaction with the reservoir. This set is symbolically denoted by I. Note that this interaction Hamiltonian admits counter-rotating terms. In this appendix, we further assume that there is no degeneracy in the transition frequencies (energy differences)

$$\omega_{mn} = E^{(0)}_m - E^{(0)}_n, \tag{A2}$$

which requires at least $h_{nn}(\boldsymbol{k}) = 0$.

A master equation is then derived from the total Hamiltonian

$$H_{\rm tot} = H^{(0)} + H_{\rm R} + H_{\rm int} \tag{A3}$$

under the usual Born–Markov approximation in the weak-coupling regime [1, 2, 3, 4, 5, 6, 7, 8] or in the van Hove limit [25, 26, 27], and reads (3.1) [8, 27]. The Hamiltonian H is still diagonalized by the same eigenstates as $H^{(0)}$ even with the Lamb shifts due to the interaction with the reservoir taken into account. The decay constants γ_{mn} are given by the on-shell form factors with the temperature effect of the reservoir included [27],

$$\gamma_{mn} = [1+N(\omega_{nm})]\Gamma_{mn}(\omega_{nm}) + N(\omega_{mn})\Gamma_{nm}(\omega_{mn}), \tag{A4}$$

where

$$\Gamma_{mn}(\omega) = 2\pi \int d^3\boldsymbol{k}\, |h_{mn}(\boldsymbol{k})|^2 \delta(\omega_{\boldsymbol{k}} - \omega), \tag{A5a}$$

$$N(\omega) = \frac{1}{e^{\beta\omega}-1}. \tag{A5b}$$

Note that $\Gamma_{mn}(\omega) = 0$ for $\omega < 0$ and that the detailed balance condition

$$\gamma_{nm} = e^{-\beta\omega_{mn}}\gamma_{mn} \tag{A6}$$

holds.

The master equation (3.1) is not the most general master equation of the Lindblad form. The characteristic feature of (3.1) is that the superoperators of the unitary part (the first term of the generator) $\mathcal{H}$ and of the dissipator (the second) $\mathcal{D}$ commute with each other, $[\mathcal{H},\mathcal{D}] = 0$. The other types of master equation are derived in different scaling limits [8, 26].

1. Two Qubits Immersed in a Bosonic Reservoir at the Zero Temperature

The master equation (2.1) appears in the dynamics of two qubits, say A and B, in interaction with a common bosonic reservoir at temperature $T = 0$. Actually, one can consider the total Hamiltonian (A3) for the system (two qubits) plus the reservoir, with the Hamiltonians

$$H^{(0)} = \frac{\Omega}{2}\Sigma_3 + g(\sigma^{(\rm A)}_+\sigma^{(\rm B)}_- + \sigma^{(\rm A)}_-\sigma^{(\rm B)}_+), \tag{A7a}$$

$$H_{\rm int} = \int d^3\boldsymbol{k}\, [h^*(\boldsymbol{k})\Sigma_+ a_{\boldsymbol{k}} + h(\boldsymbol{k})\Sigma_- a^\dagger_{\boldsymbol{k}}], \tag{A7b}$$

and (A1b), where $\boldsymbol{\Sigma} = \boldsymbol{\sigma}^{(\rm A)} + \boldsymbol{\sigma}^{(\rm B)}$ is (apart from the normalization) the total spin $\boldsymbol{S}$. $H^{(0)}$ is diagonalized in terms of the total spin $\boldsymbol{S}$ ($S = 0, 1$),

$$H^{(0)} = \Omega S_3 + g[S(S+1) - S_3^2 - 1] = \sum_{i=0,1,2,s} E^{(0)}_i |i\rangle\langle i|, \tag{A8}$$

with

$$|2\rangle = |\uparrow\rangle_{\rm A}|\uparrow\rangle_{\rm B}, \tag{A9a}$$

$$|1\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle_{\rm A}|\downarrow\rangle_{\rm B} + |\downarrow\rangle_{\rm A}|\uparrow\rangle_{\rm B}\Big), \tag{A9b}$$

$$|0\rangle = |\downarrow\rangle_{\rm A}|\downarrow\rangle_{\rm B}, \tag{A9c}$$

$$|s\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle_{\rm A}|\downarrow\rangle_{\rm B} - |\downarrow\rangle_{\rm A}|\uparrow\rangle_{\rm B}\Big), \tag{A9d}$$

and

$$E^{(0)}_2 = \Omega, \quad E^{(0)}_1 = g, \quad E^{(0)}_0 = -\Omega, \quad E^{(0)}_s = -g. \tag{A10}$$

The interaction Hamiltonian $H_{\rm int}$ is then written as

$$H_{\rm int} = \int d\boldsymbol{k}\, \sqrt{2}\, h(\boldsymbol{k}) \Big(|0\rangle\langle 1| + |1\rangle\langle 2|\Big) a^\dagger_{\boldsymbol{k}} + {\rm h.c.} \tag{A11}$$

Notice that the singlet sector is decoupled from the reservoir. The triplet sector only suffers from dissipation and is described, under the Born–Markov approximation, by the master equation (2.1), provided the decay constants for the two processes, $|2\rangle \to |1\rangle$ and $|1\rangle \to |0\rangle$, given by

$$\gamma_{12} = 2\Gamma(\Omega - g), \quad \gamma_{01} = 2\Gamma(\Omega + g), \tag{A12a}$$

$$\Gamma(\omega) = 2\pi \int d^3\boldsymbol{k}\, |h(\boldsymbol{k})|^2 \delta(\omega_{\boldsymbol{k}} - \omega), \tag{A12b}$$

are the same, $\gamma_{12} = \gamma_{01} = \gamma$. It is understood that the energy shifts, say δ_i, due to the interaction have already been included in E_i of (2.2). Actually, we know that

$$E_2 = E^{(0)}_2 + \delta_2,\ E_1 = E^{(0)}_1 + \delta_1,\ E_0 = E^{(0)}_0,\ E_s = E^{(0)}_s. \tag{A13}$$

2. Spin-Boson Model at a Finite Temperature

A master equation of the form

$$\dot{\rho}(t) = -\frac{i}{2}\Omega[\sigma_z, \rho(t)] - \gamma_0[\rho(t) - \sigma_z\rho(t)\sigma_z] - \gamma_+\left(\frac{1}{2}\{\sigma_+\sigma_-, \rho(t)\} - \sigma_-\rho(t)\sigma_+\right) - \gamma_-\left(\frac{1}{2}\{\sigma_-\sigma_+, \rho(t)\} - \sigma_+\rho(t)\sigma_-\right), \quad \text{(A14)}$$

which reduces to (2.15) when $\gamma_0 = 0$, is derived from the Hamiltonian (A3) with, for instance [27],

$$H^{(0)} = \frac{\Omega}{2}\sigma_z, \quad \text{(A15a)}$$

$$H_{\rm int} = \sigma_z \int d^3\boldsymbol{k}\,[h_0^*(\boldsymbol{k})a_{\boldsymbol{k}} + h_0(\boldsymbol{k})a_{\boldsymbol{k}}^\dagger] + \sigma_x \int d^3\boldsymbol{k}\,[h_1^*(\boldsymbol{k})a_{\boldsymbol{k}} + h_1(\boldsymbol{k})a_{\boldsymbol{k}}^\dagger], \quad \text{(A15b)}$$

and (A1b). Note that, in the above general framework for deriving (3.1), such terms like the first term of this interaction Hamiltonian $H_{\rm int}$, that do not induce transition between the two states of the spin system, are excluded, i.e. $h_{nn}(\boldsymbol{k}) = 0$. The second term of the generator of the master equation (A14) originates from this interaction and is not found in (3.1). The decay constants in the generator of (A14) and (2.15) are given by [27]

$$\gamma_0 = \beta^{-1}\Gamma_0'(0^+), \quad \text{(A16a)}$$

$$\gamma_+ = [1 + N(\Omega)]\Gamma_1(\Omega), \quad \gamma_- = N(\Omega)\Gamma_1(\Omega), \quad \text{(A16b)}$$

where

$$\Gamma_i(\omega) = 2\pi \int d^3\boldsymbol{k}\,|h_i(\boldsymbol{k})|^2\delta(\omega_{\boldsymbol{k}} - \omega) \quad (i = 0, 1). \quad \text{(A17)}$$

Between the two decay constants, the detailed balance condition holds,

$$\gamma_- = e^{-\beta\Omega}\gamma_+. \quad \text{(A18)}$$

APPENDIX B: SOLVING A MASTER EQUATION FOR A TWO-LEVEL SYSTEM

The master equation (A14) for a two-level system at a finite temperature is solved as follows. This is the master equation (2.15) when $\gamma_0 = 0$, and is derived from a spin-boson model: see Appendix A 2.

First, we arrange the master equation (A14) into

$$\dot{\rho}_{\rm I}(t) = \gamma_0\sigma_z\rho_{\rm I}(t)\sigma_z + \gamma_+\sigma_-\rho_{\rm I}(t)\sigma_+e^{-\gamma t} + \gamma_-\sigma_+\rho_{\rm I}(t)\sigma_-e^{\gamma t} \equiv (\gamma_0\mathcal{P}_z + \gamma_+e^{-\gamma t}\mathcal{P}_- + \gamma_-e^{\gamma t}\mathcal{P}_+)\rho_{\rm I}(t), \quad \text{(B1)}$$

where $\rho_{\rm I}(t)$ is defined by (2.5), i.e.

$$\rho(t) = e^{At}\rho_{\rm I}(t)e^{A^\dagger t} \equiv e^{\mathcal{A}t}\rho_{\rm I}(t) \quad \text{(B2)}$$

with

$$A = -\frac{1}{4}(\gamma^\beta + 2\gamma_0) - \frac{1}{4}(\gamma + 2i\Omega)\sigma_z, \quad \text{(B3a)}$$

$$\gamma^\beta = \gamma_+ + \gamma_-, \quad \gamma = \gamma_+ - \gamma_-, \quad \text{(B3b)}$$

and we have introduced superoperators $\mathcal{P}_z$, $\mathcal{P}_\pm$, and $\mathcal{A}$, defined by

$$\mathcal{P}_z\rho = \sigma_z\rho\sigma_z, \quad \mathcal{P}_-\rho = \sigma_-\rho\sigma_+, \quad \mathcal{P}_+\rho = \sigma_+\rho\sigma_-, \quad \text{(B4a)}$$

$$\mathcal{A}\rho = A\rho + \rho A^\dagger. \quad \text{(B4b)}$$

These operators $\mathcal{P}_i$'s satisfy

$$\mathcal{P}_z^2 = 1, \quad \mathcal{P}_\pm^2 = 0, \quad \mathcal{P}_z\mathcal{P}_\pm = \mathcal{P}_\pm = \mathcal{P}_\pm\mathcal{P}_z, \quad \text{(B5a)}$$

$$\mathcal{P}_-\mathcal{P}_+\mathcal{P}_- = \mathcal{P}_-, \quad \mathcal{P}_+\mathcal{P}_-\mathcal{P}_+ = \mathcal{P}_+. \quad \text{(B5b)}$$

Now define $\tilde{\rho}_{\rm I}(t) = e^{-\gamma_0\mathcal{P}_z t}\rho_{\rm I}(t)$, then it satisfies

$$\frac{d}{dt}\tilde{\rho}_{\rm I}(t) = (\gamma_+e^{-\gamma t}\mathcal{P}_- + \gamma_-e^{\gamma t}\mathcal{P}_+)\tilde{\rho}_{\rm I}(t). \quad \text{(B6)}$$

This equation yields the following equations [notice the relations (B5)]:

$$\frac{d}{dt}\begin{pmatrix} e^{-\gamma t}\mathcal{P}_-\tilde{\rho}_{\rm I}(t) \\ \mathcal{P}_-\mathcal{P}_+\tilde{\rho}_{\rm I}(t)\end{pmatrix} = \begin{pmatrix} -\gamma & \gamma_- \\ \gamma_+ & 0\end{pmatrix}\begin{pmatrix} e^{-\gamma t}\mathcal{P}_-\tilde{\rho}_{\rm I}(t) \\ \mathcal{P}_-\mathcal{P}_+\tilde{\rho}_{\rm I}(t)\end{pmatrix}, \quad \text{(B7a)}$$

$$\frac{d}{dt}\begin{pmatrix} e^{\gamma t}\mathcal{P}_+\tilde{\rho}_{\rm I}(t) \\ \mathcal{P}_+\mathcal{P}_-\tilde{\rho}_{\rm I}(t)\end{pmatrix} = \begin{pmatrix} \gamma & \gamma_+ \\ \gamma_- & 0\end{pmatrix}\begin{pmatrix} e^{\gamma t}\mathcal{P}_+\tilde{\rho}_{\rm I}(t) \\ \mathcal{P}_+\mathcal{P}_-\tilde{\rho}_{\rm I}(t)\end{pmatrix}, \quad \text{(B7b)}$$

which are easily solved to give

$$e^{-\gamma t}\mathcal{P}_-\tilde{\rho}_{\rm I}(t) = \frac{1}{\gamma^\beta}\left[\frac{d}{dt}(e^{\gamma_- t} - e^{-\gamma_+ t})\mathcal{P}_- + \gamma_-(e^{\gamma_- t} - e^{-\gamma_+ t})\mathcal{P}_-\mathcal{P}_+\right]\rho(0), \quad \text{(B8a)}$$

$$e^{\gamma t}\mathcal{P}_+\tilde{\rho}_{\rm I}(t) = \frac{1}{\gamma^\beta}\left[\frac{d}{dt}(e^{\gamma_+ t} - e^{-\gamma_- t})\mathcal{P}_+ + \gamma_+(e^{\gamma_+ t} - e^{-\gamma_- t})\mathcal{P}_+\mathcal{P}_-\right]\rho(0). \quad \text{(B8b)}$$

Inserting these results into (B6) and integrating its right-hand side, we obtain

$$\tilde{\rho}_{\rm I}(t) = \rho(0) + \frac{1}{\gamma^\beta}[\gamma_+(e^{\gamma_- t} - e^{-\gamma_+ t})\mathcal{P}_- + (\gamma_+e^{\gamma_- t} + \gamma_-e^{-\gamma_+ t} - \gamma^\beta)\mathcal{P}_-\mathcal{P}_+]\rho(0) + \frac{1}{\gamma^\beta}[\gamma_-(e^{\gamma_+ t} - e^{-\gamma_- t})\mathcal{P}_+ + (\gamma_-e^{\gamma_+ t} + \gamma_+e^{-\gamma_- t} - \gamma^\beta)\mathcal{P}_+\mathcal{P}_-]\rho(0), \quad \text{(B9)}$$

which, together with the relations (B5) and $e^{\gamma_0 \mathcal{P}_z t} = \cosh\gamma_0 t + \mathcal{P}_z \sinh\gamma_0 t$, yields

$$\begin{aligned}\rho_{\rm I}(t) = (\cosh\gamma_0 t + \mathcal{P}_z \sinh\gamma_0 t)\rho(0) \\ + \frac{e^{\gamma_0 t}}{\gamma^\beta}[\gamma_+(e^{\gamma_- t} - e^{-\gamma_+ t})\mathcal{P}_- \\ + (\gamma_+ e^{\gamma_- t} + \gamma_- e^{-\gamma_+ t} - \gamma^\beta)\mathcal{P}_-\mathcal{P}_+]\rho(0) \\ + \frac{e^{\gamma_0 t}}{\gamma^\beta}[\gamma_-(e^{\gamma_+ t} - e^{-\gamma_- t})\mathcal{P}_+ \\ + (\gamma_- e^{\gamma_+ t} + \gamma_+ e^{-\gamma_- t} - \gamma^\beta)\mathcal{P}_+\mathcal{P}_-]\rho(0).\end{aligned} \tag{B10}$$

In order to reach the solution $\rho(t)$, the following relations suffice

$$e^{\mathcal{A}t}\mathcal{P}_- = e^{-(\gamma_- + \gamma_0)t}\mathcal{P}_-, \qquad e^{\mathcal{A}t}\mathcal{P}_+ = e^{-(\gamma_+ + \gamma_0)t}\mathcal{P}_+ \tag{B11}$$

to give

$$\begin{aligned}\rho(t) = (\cosh\gamma_0 t + \mathcal{P}_z \sinh\gamma_0 t)e^{\mathcal{A}t}\rho(0) + \frac{1}{\gamma^\beta}[\gamma_+(1 - e^{-\gamma^\beta t})\mathcal{P}_- + (\gamma_+ + \gamma_- e^{-\gamma^\beta t} - \gamma^\beta e^{-\gamma_- t})\mathcal{P}_-\mathcal{P}_+]\rho(0) \\ + \frac{1}{\gamma^\beta}[\gamma_-(1 - e^{-\gamma^\beta t})\mathcal{P}_+ + (\gamma_- + \gamma_+ e^{-\gamma^\beta t} - \gamma^\beta e^{-\gamma_+ t})\mathcal{P}_+\mathcal{P}_-]\rho(0).\end{aligned} \tag{B12}$$

This reduces to (2.19) when $\gamma_0 = 0$.

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